Problem. Let $c > 0$ be a given positive real and $\mathbb{R}_{>0}$ be the set of all positive reals. Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f((c+1)x + f(y)) = f(x+2y) + 2cx \quad \text{for all } x, y \in \mathbb{R}_{>0}.$$

After trying to solve the problem on your own, you can find a possible solution on the next page.

This problem is related to functional equations. Check out the skillpage[a] to help you solve the problem.

[a] https://calimath.org/skillpages/functional-equations

Problem. Let $c > 0$ be a given positive real and $\mathbb{R}_{>0}$ be the set of all positive reals. Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f((c+1)x + f(y)) = f(x+2y) + 2cx \quad \text{for all } x, y \in \mathbb{R}_{>0}.$$

Proof. We can see that $f(x) = 2x$ satisfies the given condition. Now, we want to prove that this is the only solution. Let $P(x, y)$ denote the assertion that

$$f((c+1)x + f(y)) = f(x+2y) + 2cx.$$

Assume we find some z with $f(z) < 2z$. $P(\frac{2z-f(z)}{c}, z)$ gives

$$f(2z + \frac{2z - f(z)}{c}) = f(2z + \frac{2z - f(z)}{c}) + 2cz.$$

From $2cz > 0$, we get a contradiction. Thus,

$$f(x) \geq 2x \quad \forall x \in \mathbb{R}_{>0} \tag{1}$$

Now, assume that $f(z) > 2z$ for some positive real z. Thus, we find a positive real number d satisfying $f(z) = 2z + d$. We want to prove by induction that $f(x) \geq 2x + k \cdot d$ for every integer $k \geq 0$ and every real number $x > 2z$. The base case follows from eq. (1). For the induction step, assume $f(x) \geq 2x + kd$ for every $x > 2z$. $P(a, z)$ yields

$$\begin{aligned} f(a+2z) &= f((c+1)a + f(z)) - 2ca \\ &= f((c+1)a + 2z + d) - 2ca \\ &\geq 2((c+1)a + 2z + d) + kd - 2ca \\ &= 2a + 4z + (k+2)d \\ &> 2(a+2z) + (k+1)d \end{aligned}$$

Since a can be an arbitrary positive integer, our claim follows. Now take $k > \frac{f(2z+1)}{d}$. We get

$$f(2z+1) \geq 4z + 2 + k \cdot d > k \cdot d > \frac{f(2z+1)}{d} \cdot d = f(2z+1).$$

This is a contradiction. Thus, $f(x) = 2x$ for every $x \in \mathbb{R}_+$ follows. q.e.d.