Problem. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) + xf(y) = f(xy + y) + f(x)$$

for reals x, y.

After trying to solve the problem on your own, you can find a possible solution on the next page.

This problem is related to functional equations. Check out the skillpage[a] to help you solve the problem.

[a] https://calimath.org/skillpages/functional-equations

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Proof. Let $P(x,y)$ denote the given functional equation. We can see that the function $f(x)=0$ satisfies the given condition. We want to figure out if there are any other solutions. So from now on, assume that we find an a with $f(a)\neq 0$. $P(0,0)$ gives

$$f(f(0))=2f(0).$$

If $f(0)\neq 0$, we get $f(f(0))\neq 0$. So, w.l.o.g assume $a\neq 0$. For $x\neq 0$, $P\left(x,\frac{f(x)}{x}\right)$ yields

$$f\left(f(x)+\frac{f(x)}{x}\right)+xf\left(\frac{f(x)}{x}\right)=f\left(f(x)+\frac{f(x)}{x}\right)+f(x)\quad \forall x\neq 0.$$

We conclude

$$f\left(\frac{f(x)}{x}\right)=\frac{f(x)}{x}\quad \forall x\neq 0. \tag{1}$$

Using $a\neq 0$ and plugging in $x=\frac{f(a)}{a}$ yields

$$f\left(\frac{f\left(\frac{f(a)}{a}\right)}{\frac{f(a)}{a}}\right)=\frac{f\left(\frac{f(a)}{a}\right)}{\frac{f(a)}{a}}.$$

With $f(a)\neq 0$ and eq. (1), we conclude $f(1)=1$. $P(0,y)$ gives

$$f(f(0)+y)=f(y)+f(0). \tag{2}$$

Claim 1. $f(0)=0$.

Proof. Assume $f(0)\neq 0$. With $x=f(0)$ into eq. (1), we get $f(2)=2$. Let b and c be real numbers with $f(b)=f(c)$. With $P(b,0)$ and $P(c,0)$, we get

$$f(f(b))+bf(0)=f(0)+f(b)\quad\text{and}\quad f(f(c))+cf(0)=f(0)+f(c).$$

$f(0)\neq 0$ implies $b=c$ and thus, f is injective. $b=-1$ gives

$$f(f(-1))=2f(0)+f(-1)\overset{\text{eq. (2)}}{=}f(-1+f(0))+f(0)\overset{\text{eq. (2)}}{=}f(-1+2f(0)).$$

From f injective, we conclude $f(-1)=2f(0)-1$. $P(-1,y)$ yields

$$f(f(-1)+y)-f(y)=f(0)+f(-1).$$

With

$$f(f(-1)+y)=f(-1+2f(0)+y)\overset{\text{eq. (2)}}{=}f(y-1)+2f(0),$$

we conclude

$$f(y-1)-f(y)=-f(0)+f(-1).$$

Setting $y=1$ and $y=2$ and comparing the left hand sides gives $f(0)-f(1)=f(1)-f(2)$. We conclude $f(0)=0$ as a contradiction. □

$P(x, 0)$ gives

$$f(f(x)) = f(x).$$

$P(-1, y)$ gives

$$f(f(-1) + y) = f(-1) + f(y).$$

Thus, we get

$$\begin{aligned} f(x) + f(-1) + xf(-1) &= f(f(x)) + f(-1) + xf(-1) \\ &= f(f(x) + f(-1)) + xf(f(-1)) \\ &\overset{P(x,f(-1))}{=} f(xf(-1) + f(-1)) + f(x). \end{aligned}$$

If $f(-1) \neq 0$, we take $x = \frac{y}{f(-1)} - 1$ to get

$$f(y) = y \quad \forall y \in \mathbb{R}.$$

We can check that this is indeed a solution.
If $f(-1) = 0$, $P(1, -1)$ gives

$$f(1 - 1) + f(-1) = f(-2) + f(1),$$

which implies $f(-2) = -1$. Now, $P(-2, 1)$ yields

$$-2 = f(-1 + 1) - 2f(1) = f(-1) + f(-2) = -1.$$

This is a contradiction, so we have no solutions in this case, and hence, we are done. q.e.d.