Problem. In triangle ABC, $AB = BC$, and let I be the incentre of $\triangle ABC$. M is the midpoint of segment BI. P lies on segment AC, such that $AP = 3PC$. H lies on line PI, such that $MH \perp PH$. Q is the midpoint of the arc AB of the circumcircle of $\triangle ABC$. Prove that $BH \perp QH$.

After trying to solve the problem on your own, you can find a possible solution on the next page.

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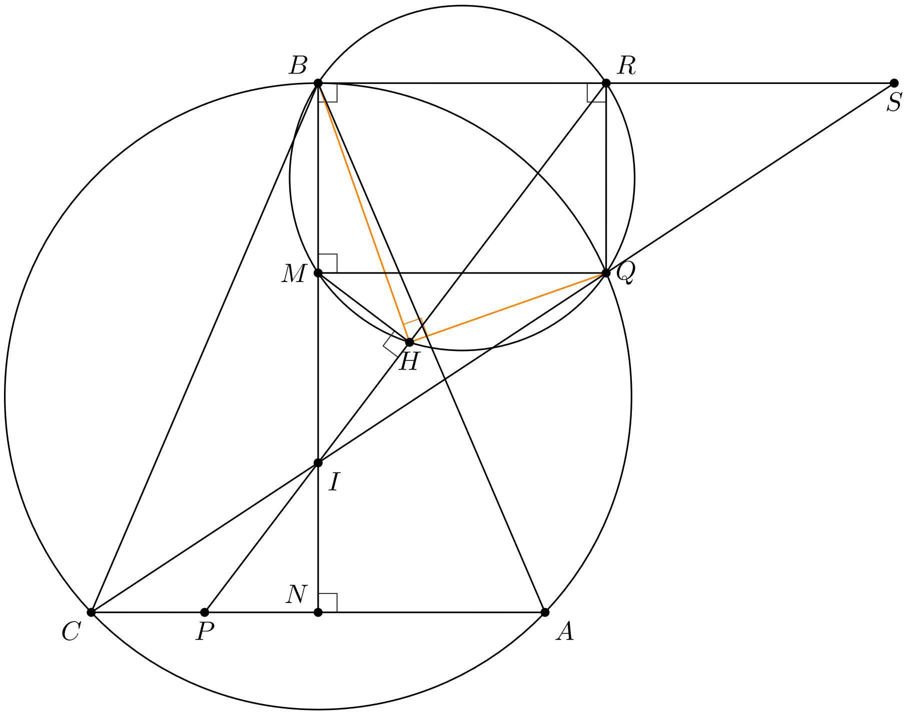

Proof. We use directed angles mod 180°.[1] Let R be the second intersection of PI with the circumcircle of BMH beside H and let S be the intersection of BR with CI. Moreover, let N be the midpoint of AC. This means $AC \perp BN$ since ABC is isosceles.

Since

$$\measuredangle NBR = \measuredangle MBR = \measuredangle RHM = \measuredangle MHP = 90^\circ,$$

we get that lines AC and BR are parallel. Moreover, BR is the exterior angel bisector of $\measuredangle CBA$ and since CI is the interior angle bisector of $\measuredangle ACB$, we get that S is the C-excenter of triangle ABC. Thus by the incenter/excenter lemma[2] we get

$$QA = QB = QI = QS. \tag{1}$$

Since $AP = 3PC$ we know that P is the midpoint of AN. Now since AC and BS are parallel R is the midpoint of BS. We also know that m is the midpoint of BI by definition. Because of 1 we get that QM is perpendicular to BI and QR is perpendicular to BS. Therefore the points $MBRQ$ lie on the circle with diameter BQ. Now since $HMBR$ is concyclic, H must also lie on the circle with diameter BQ. Thus $\measuredangle QHB = 90^\circ$, which gives the desired result.

q.e.d.

[1] https://calimath.org/wiki/directed-angles
[2] https://calimath.org/wiki/incenter-excenter-lemma