Problem. A number n is interesting if 2018 divides $d(n)$ (the number of positive divisors of n)[a]. Determine all positive integers k such that there exists an infinite arithmetic progression with common difference k whose terms are all interesting.

[a] https://calimath.org/wiki/divisor-sum-function

After trying to solve the problem on your own, you can find a possible solution on the next page.

This problem is related to number theoretic functions. Check out the skillpage[a] to help you solve the problem.

[a] https://calimath.org/skillpages/number-theoretic-functions

Problem. A number n is interesting if 2018 divides $d(n)$ (the number of positive divisors of n)[a]. Determine all positive integers k such that there exists an infinite arithmetic progression with common difference k whose terms are all interesting.

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Proof. We will prove that the solutions are all positive integers k such that there exists a prime p with $\nu_p(k) \geq 1009$ and $k \neq 2^{1009}$. Let $a_n = s + kn$ be the nth element in our arithmetic progression.
At first, we give a construction for these values of k. We can write $k = p^{1009} \cdot t$ with a positive integer t for p odd and a positive integers $t > 1$ for $p = 2$. We want to choose $s = p^{1008} \cdot m$, with $p \nmid m$. Then

$$a_n = s + kn = p^{1008} \cdot (m + pt)$$

since $\gcd(p^{1008}, m + pt) = 1$ and since d is multiplicative, we have $d(a_n) = d(p^{1008}) \cdot d(m+pt) = 1009 \cdot d(m + pt)$. So we want that 2 divides $d(m + pt)$. This holds if and only if $m + pt$ is not a perfect square. If $p > 2$, we have exactly $\frac{p+1}{2} < \frac{p+p}{2} = p$ quadratic residues mod p[1]. Since 0 is a quadratic residue, we can choose $m \in \{1, 2, \dots, p-1\}$ such that m is not a quadratic residue mod p. Thus, $m + pt$ can't be a perfect square, and we are done. Now let's consider the case $p = 2$. If we find a prime $q > 2$ dividing t, we can find $m' \in \{1, \dots, q-1\}$ which is not a quadratic residue mod q. Now we choose $m = m'$ if m' is odd and $m = m' + q$, if m' is even. Thus, m is odd and $m + pt$ can't be a perfect square, and we are done. If t is a power of 2, we know $4 \mid t$. Thus, we can choose $m = 3$ which is not a quadratic residue mod 4. Thus $m + pt$ can't be a perfect square, and we are done again.

Now we went to prove that for every other k and an arithmetic progression with common difference k, we can find an element a_n with $2018 \nmid d(a_n)$. For the sake of contradiction, assume that $a_n = s + kn$ satisfies $2018 \mid d(n)$ for every positive integer n. Let $g := \gcd(s, k)$, $s' := \frac{s}{g}$ and $k' = \frac{k}{g}$. So we have

$$a_n = g(s' + k'n),$$

with coprime integers s' and k'. By Dirichlet's theorem[2], there are infinitely many integers n such that $s' + k'n$ is a prime number. So, we can find an n' such that $s' + k'n' =: q$ is a prime with $q \nmid g$. Thus, $d(n') = d(g) \cdot 2$. So

$$1009 \mid d(g) = \prod_{p \text{ prime}} (\nu_p(g) + 1).$$

1009 is a prime number, and therefore we can find a prime number p_0 with $\nu_{p_0}(g) \equiv -1 \mod 1009$. Since $\nu_p(g) \leq \nu_p(k) \leq 1009$ for every prime number p, we have $\nu_{p_0}(g) = 1008$.
At first, consider the case $k = 2^{1009}$. Here we must have $g = 2^{1008}$ and

$$a_n = 2^{1008} \cdot (s' + 2n)$$

Let x^2 be a perfect square greater than s' with the same parity as s'. choosing $n = \frac{x^2 - s'}{2}$ gives

$$a_n = 2^{1008} x^2.$$

Thus, $d(a_n)$ is not divisible by 2 ,a contradiction.
We now assume $k \neq 2^{1009}$. Thus, $\nu_p(k) \leq 1008$. We denote with $p_1, p_2, \dots, p_l$ the prime numbers with $\nu_{p_i}(g) = 1008$. Let $P = p_1 \cdot p_2 \cdot \ldots \cdot p_l$. We know that $p_i \nmid k'$ since k is at least 1008 times divisible by p_i. Therefore, there exists an multiplicative inverse of $k' \mod P$.So we find $r \in \{1, 2, \dots, P\}$ with $r \equiv \frac{-s'}{k'} \mod P$. This implies $s' + k'r \equiv 0 \mod P$. Let's define $n_m := r + Pm$. From now on, we only want to consider the terms

$$a_{n_m} := a_{r+Pm} = g \cdot (s' + k'(r + Pm)) = g \cdot ((s' + k'r) + k'Pm) = gP \cdot (\frac{s' + k'r}{P} + k'm)$$

[1] https://calimath.org/wiki/legendre-symbol
[2] https://calimath.org/wiki/dirichlets-theorem-(primes)

From $p_i \nmid k'$, we get

$$\gcd\left(\frac{s'+k'r}{P}, k'\right) = \gcd(s'+k'r, k') = \gcd(s', k') = 1$$

Thus, by Dirchtlet's theorem, we can find a prime $q \nmid gP$ and a positive integer m' such that $\frac{s'+k'r}{P} + k'm' = q$. Hence,

$$\begin{aligned} d(a_{n_{m'}}) &= d(gP) \cdot 2 \\ &= 2 \cdot \prod_{i=1}^{l} (\nu_{p_i}(gP) + 1) \cdot \prod_{p \notin \{p_1, \ldots, p_l\} \text{prime}} (\nu_p(gP) + 1) \\ &= 2 \cdot 1010^l \cdot \prod_{p \notin \{p_1, \ldots, p_l\} \text{prime}} (\nu_p(g) + 1) \end{aligned}$$

Since $\nu_p(g) < 1008$ for every prime $p \notin \{p_1, \ldots, p_l\}$ we conclude $1009 \nmid d(n_{m'})$.

q.e.d.