Problem. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following condition: for any real numbers x and y, the number $f(x + f(y))$ is equal to $x + f(y)$ or $f(f(x)) + y$

After trying to solve the problem on your own, you can find a possible solution on the next page.

This problem is related to functional equations. Check out the skillpage[a] to help you solve the problem.

[a] https://calimath.org/skillpages/functional-equations

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Proof. We can directly see that the function $f(x)=x$ satisfies the given condition. From now on assume that we can find $a\in\mathbb{R}$ such that $f(a)\neq a$.
Inserting $x-f(y)$ for x in our given condition implies

$$f(x)\in\{x,f(f(x-f(y)))+y\}.$$

Let us denote this assertion by $P(x,y)$.
By $P(a,y)$ we get $f(a)=f(f(a-f(y)))+y$ for all $y\in\mathbb{R}$. In the same way $P(a,f(f(a-f(y))))$ implies

$$f(f(a-f(f(f(a-f(y))))))+f(f(a-f(y)))=f(a)=f(f(a-f(y)))+y.$$

This yields

$$f(f(a-f(f(f(a-f(y))))))=y.$$

Hence, f is bijective.
$P(x,0)$ gives us

$$f(x)\in\{x,f(f(x-f(0)))\}.$$

In the case that $f(x)\neq x$, we have $f(x)=f(f(x-f(0)))$. With the injectivity of f, this would imply $x=f(x-f(0))$.
In total, this gives us

$$x\in\{f(x),f(x-f(0))\}.$$

Taking $f(x)$ for x, we get

$$f(x)\in\{f(f(x)),f(f(x)-f(0))\}.$$

Using the injectivity of f, this implies

$$x\in\{f(x),f(x)-f(0)\},$$

or

$$f(x)\in\{x,x+f(0)\}.$$

We get $a\neq f(a)=a+f(0)$, which implies $f(0)=:c\neq 0$. Assume that we can find $b\in\mathbb{R}$ such that $f(b)=b$. Then $P(a,b)$ gives us

$$a+c=f(a)=f(f(a-f(b)))+b=f(f(a-b))+b.$$

For $f(a-b)=a-b$, the right-hand side is equal to a, a contradiction to $c\neq 0$. So, $f(a-b)=a-b+c$. Since f is bijective, this implies $f(a-b+c)=a-b+2c$. Hence, $f(f(a-b))+b=a+2c\neq a+c$, a contradiction as well. So, $f(x)=x+c$ for all $x\in\mathbb{R}$. We see that in this case $f(x+f(y))=f(f(x))+y$ holds for all $x,y\in\mathbb{R}$. Hence, $f(x)=x$ and $f(x)=x+c$ for some $c\in\mathbb{R}$ are the only solutions. q.e.d.