Problem. Find all the pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

After trying to solve the problem on your own, you can find a possible solution on the next page.

This problem is related to number theoretic functions. Check out the skillpage[a] to help you solve the problem.

[a] https://calimath.org/skillpages/number-theoretic-functions

Problem. Find all the pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

Proof. We see that for $x = 1$ and p arbitrary, the condition is satisfied. Moreover, for $p = 2$, $(p-1)^x + 1 = 2$, and thus, $x \in \{1, 2\}$. So, we also have the solution $(x, p) = (2, 2)$. From now on, assume $p > 2$ and $x > 1$.

From $p > 2$, we conclude that p is odd, and therefore $(p-1)^x + 1$ is also odd. We get that x must be odd as well. Let $q \geq 3$ be the smallest prime divisor of x. Let $d := ord_q(p-1)$ be the order[1] of $p-1$ modulo q. We know $q \mid (p-1)^x + 1$ and thus $(p-1)^{2x} \equiv 1 \mod q$. Thus, $d \mid 2x$, and by Fermat's little theorem[2], we also have $d \mid q-1$. This yields

$$d \mid \gcd(q-1, 2x).$$

Since q is the smallest prime dividing x, we know $\gcd(q-1, x) = 1$, and from q odd, we conclude $d = 2$. Thus,

$$q \mid (p-1)^2 - 1 = p(p-2).$$

Since q is a prime number, we have $q \mid p$ or $q \mid p-2$. In the second case, we get

$$0 \equiv (p-1)^x + 1 \equiv 1^x + 1 \equiv 2 \mod q.$$

This is a contradiction to $q \geq 3$. Therefore, we have $q \mid p$, and since p is prime, we conclude $q = p$. We have $q \mid x \leq 2p = 2q$. This is only possible for $x = q$ or $x = 2q$. x odd implies $x = q$. Since $q \mid (p-1) + 1$ and q is odd, we can use the Lifting-the-exponent lemma[3] to get

$$\nu_q((p-1)^x + 1) = \nu_q(p-1+1) + \nu_q(x) = 1 + 1 = 2.$$

We also have

$$v_p(p^{(p-1)}) = p - 1 = q - 1$$

and thus, $q - 1 \leq 2$. This is only possible for $x = p = q = 3$. We indeed see that

$$3^{3-1} = 9 \mid 9 = (3-1)^3 + 1.$$

So, $(x, p) = (3, 3)$ is our last solution. q.e.d.

[1] https://calimath.org/wiki/multiplicative-order
[2] https://calimath.org/wiki/fermats-little-theorem
[3] https://calimath.org/wiki/lifting-the-exponent-lemma