Problem. Let n be a positive integer. A Japanese triangle consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A ninja path in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row.
In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

After trying to solve the problem on your own, you can find a possible solution on the next page.

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Proof. For any $i \leq n$, we can write $i = 2^a + b, 0 \leq b < 2^a$ and color the $2b+1(\leq i)$-th circle in the i-th row red. In this japanese triangle, any ninja path may contain at most 1 red circle in the rows $2^a, 2^a+1, \ldots, 2^{a+1}-1$. Hence, the total number of red circles in any ninja path is at most $\lceil \log_2(n+1) \rceil$. We conclude that $k \leq \lceil \log_2(n+1) \rceil$.

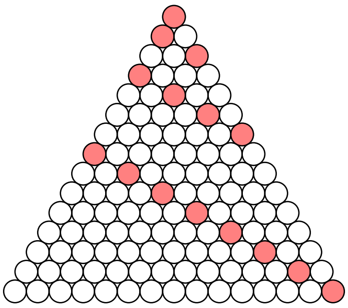

Figure 1: Upper bound

Now, consider any japanase triangle with n rows. Consider the j-th circle c in the i-th row ($j \leq i \leq n$). If the red circle in row i is in position $r \geq j$, draw an arrow from c to the bottom left. If the red circle in row i is in position $r \leq j$, draw an arrow from c to the bottom right. In this way, we draw two arrows from any red circle and one arrow from every other circle.

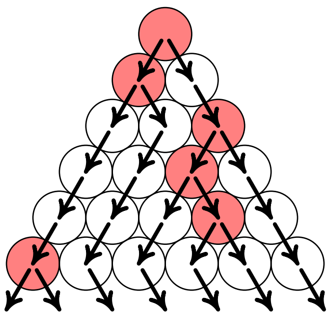

Figure 2: Lower bound

For convenience, we imagine there was an $(n+1)$-th row C_{n+1} (without a red circle). For each circle $c \in C_{n+1}$, there is a unique ninja path $p(c)$ to c that follows the arrows since any circle except for the top one has exactly one incoming arrow. We now choose a random ninja path $P \in M = \{p(c) \mid c \in C_{n+1}\}$. Start at the top circle. If you are currently on a red circle, go to one of the outgoing arrows with equal probability $1/2$. Otherwise, just follow the unique outgoing arrow. For $p \in M$, let $r(p)$ be the number of red circles in p. We have $\mathbb{P}[P = p] = 2^{-r(p)}$. Hence,

$$\mathbb{E}[2^{r(P)}] = \sum_{p \in M} 2^{r(p)} \cdot \mathbb{P}[P = p] = \sum_{p \in M} 1 = n + 1.$$

This tells us that we can find a particular path $p \in M$ such that $2^{r(p)} \geq n+1$ or $r(p) \geq \log_2(n+1)$. Hence, $k \geq \log_2(n+1)$. We conclude that $k = \lceil \log_2(n+1) \rceil$. q.e.d.

Remark 1. My first idea gave a weaker bound and I want to present it as motivation: The goal is to define a random ninja path in such a way that we hit every circle in the i-th row with equal probability $1/i$. There is a unique well-known way of achieving this. We start at the top circle (with probability $1 = 1/1$). If we are at circle j of the i-th row ($j \le i \le n$), go to the bottom left with probability $\frac{(i+1)-j}{i+1}$ and to the bottom right with probability $\frac{j}{i+1}$. By induction, the probability of passing through circle j of the $(i+1)$-th row is $\frac{1}{i}\frac{(j-1)}{i+1} + \frac{1}{i}\frac{(i+1)-j}{i+1} = \frac{1}{i+1}$, as desired. Let R_i be 1 if we visit the red circle in row i and 0 otherwise. Hence, $R = \sum_{i=1}^{n} R_i$ is the number of red circles on the random ninja path. We have $\mathbb{E}[R_i] = \mathbb{P}[\text{we visit the red circle in row } i] = 1/i$. Hence, by linearity of expectation, we expect to visit $\mathbb{E}[R] = \sum_{i=1}^{n} \mathbb{E}[R_i] = H_n \approx \log n = \ln 2 \cdot \log_2 n$ red circles. Therefore, there exists one ninja path visiting at least that many red circles and hence $k \ge H_n$.

How does this approach relate to the above solution? Motivated by the logarithmic lower-bound, I found the construction of $k = \lceil \log_2(n+1) \rceil$ right away and was pretty confident that that should be the optimal value. Hence, I tried to improve my lower-bound. I first tried to calculate $\mathbb{E}[2^R]$ to put more weight on the paths with many red circles trying to prove that $\mathbb{E}[2^R] \ge n+1$ ($> \frac{1}{2} \cdot 2^{\lceil \log_2(n+1) \rceil}$ would be enough but is a weird bound). I found out that for any probability space of ninja paths there was a japanese triangle such that $\mathbb{E}[2^R] \le 2^{H_n} \approx n^{\ln 2}$ (Why?). Thus, I tried to make the probability space dependent on the choice of japanese triangle. I also knew that in the example
Note: I didn't want to make choices depending on future rows because if there was some oracle that could tell us wether we should prefer going right or left to visit as many red circles as possible in the future, then we could just use it to directly find the best ninja path. Moreover, when searching for good paths to consider I noted that in fig. 1, $R \le \log_2(n+1)$ and hence in order to achieve $\mathbb{E}[2^R] \ge n+1$, we would need $R = \log_2(n+1)$ for any path that we choose with positive probability.
It is enough to increase our expectation $\mathbb{E}[2^R]$ by 1 when going from i to $i+1$ rows in the sense that for $n = 1$, $\mathbb{E}[2^R] = 2$ is clear and that we only want $\mathbb{E}[2^R] \ge n+1$. We know that 2^R will stay the same for all ninja paths not visiting the red circle of row $(i+1)$. As a (massively) simplifying assumption (A), consider the case where exactly one possible ninja path visits that circle and let r be the number of red circles in rows 1 to i on it. Hence, the red circle in row $(i+1)$ contributes $2^{r+1} - 2^r = 2^r$ to 2^R of that particular path. Hence, we require a probability of at least $1 \cdot (2^r)^{-1}$ of choosing that particular path. This tells us that we are "allowed" to make choices with probabilities 1/2 at every red circle but we "need to" make a deterministic choice everywhere else (to achieve exactly the bound we need). If we can make sure that (A) holds at every circle, we will be done. The probability distribution of paths described in the solution is the distribution satisfying these properties.