Problem. Determine all real numbers α such that, for every positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n. (Note that $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.)

After trying to solve the problem on your own, you can find a possible solution on the next page.

Problem. Determine all real numbers α such that, for every positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n. (Note that $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.)

Proof. We will prove that the solutions for α are all multiples of 2. Let $\alpha = 2k + r$ where $k \in \mathbb{Z}$ and $0 \leq r < 2$.

$$\begin{aligned}
\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor &= \lfloor 2k + r \rfloor + \lfloor 4k + 2r \rfloor + \cdots + \lfloor 2nk + nr \rfloor \\
&= 2k + 4k + \cdots + 2nk + \lfloor r \rfloor + \lfloor 2r \rfloor + \cdots + \lfloor nr \rfloor \\
&= n(n+1)k + \lfloor r \rfloor + \lfloor 2r \rfloor + \cdots + \lfloor nr \rfloor.
\end{aligned}$$

Hence, $\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$ is divisible by n if and only if $\lfloor r \rfloor + \lfloor 2r \rfloor + \cdots + \lfloor nr \rfloor$ is divisible by n. Thus, it is sufficient to consider $\alpha \in [0, 2)$. We easily see that $\alpha = 0$ is a solution. Now assume $\alpha \in (0, 2)$. We will show that α is not a solution in this case, which implies the stated result.

Let $f(n) = \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$. Using the inequality $x - 1 < \lfloor x \rfloor \leq x$ for each summand in α, we get

$$\alpha \frac{n(n+1)}{2} - n < f(n) \leq \alpha \frac{n(n+1)}{2}.$$

There are exactly n consecutive integers k such that $f(n) = k$ would satisfy these given two inequalities. Since $n \mid f(n)$, we conclude

$$f(n) = n \cdot \left\lfloor \frac{\alpha(n+1)}{2} \right\rfloor.$$

If we find n with $\left\lfloor \frac{\alpha(n+1)}{2} \right\rfloor = \left\lfloor \frac{\alpha(n+2)}{2} \right\rfloor$, we would have

$$\lfloor (n+1)\alpha \rfloor = f(n+1) - f(n) = (n+1) \cdot \left\lfloor \frac{\alpha(n+1)}{2} \right\rfloor - n \cdot \left\lfloor \frac{\alpha(n+1)}{2} \right\rfloor = \left\lfloor \frac{\alpha(n+1)}{2} \right\rfloor.$$

If we also have $n + 1 \geq \frac{2}{\alpha}$, then $(n+1)\alpha - \frac{\alpha(n+1)}{2} = \frac{\alpha(n+1)}{2} > 1$, which contradicts the previous equality.
Thus, it is left to prove the following claim.

Claim 1. We can find an integer $n \geq \frac{2}{\alpha} - 1$ such that $\left\lfloor \frac{\alpha(n+1)}{2} \right\rfloor = \left\lfloor \frac{\alpha(n+2)}{2} \right\rfloor$.

Take integers $m = \left\lceil \frac{2}{\alpha} \right\rceil$ and k large enough (we can take $k > \frac{2}{2-\alpha}$ to get $\frac{k\alpha}{2} < k - 1$). Then we have

$$\left\lfloor \frac{\alpha(m+k+1)}{2} \right\rfloor - \left\lfloor \frac{\alpha(m+1)}{2} \right\rfloor \leq \frac{\alpha(m+k+1)}{2} - \frac{\alpha(m+1)}{2} + 1 = \frac{\alpha k}{2} + 1 < k.$$

Hence the $k+1$ numbers $\left\lfloor \frac{\alpha(m+1)}{2} \right\rfloor, \left\lfloor \frac{\alpha(m+1+1)}{2} \right\rfloor, \ldots, \left\lfloor \frac{\alpha(m+k+1)}{2} \right\rfloor$ can only attain k different values, which implies, that two of them are equal. So, we find $0 \leq k' < k$ with $\left\lfloor \frac{\alpha(m+k'+1)}{2} \right\rfloor = \left\lfloor \frac{\alpha(m+k'+2)}{2} \right\rfloor$. Taking $n = m + k' + 1$ gives a contradiction in the way presented above. q.e.d.