Problem. Let ABC be a triangle with $AB < AC < BC$. Let the incenter and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively.
Prove that $\angle KIL + \angle YPX = 180°$.

After trying to solve the problem on your own, you can find a possible solution on the next page.

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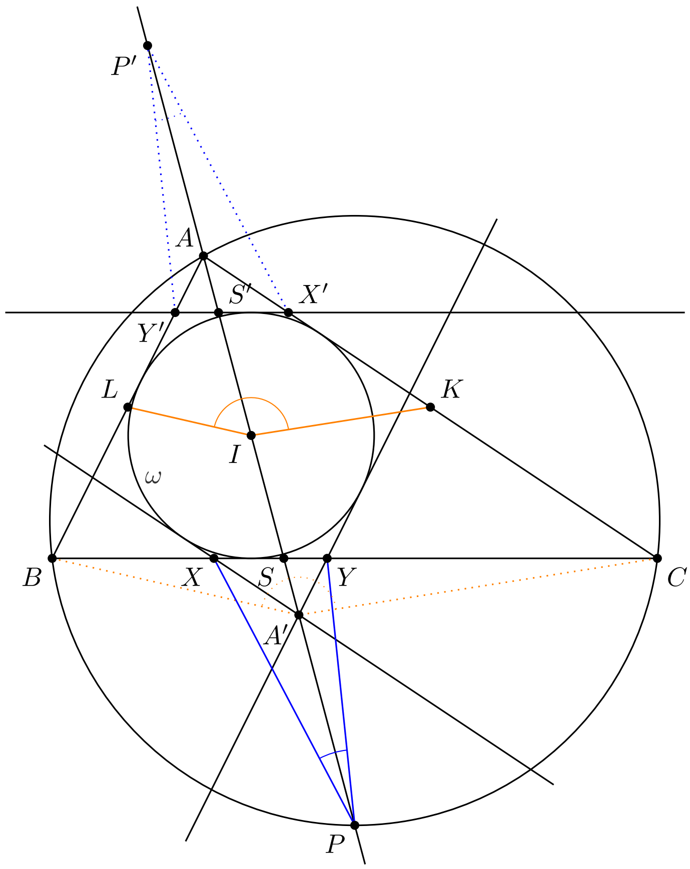

Proof. Let A', P', X', Y' be the reflections of A, P, X, Y at I. The homothety with center A and factor 2 sends I, K, L to A', C, B. So,

$$\angle KIL = \angle CA'B.$$

X', Y' lie on AC and AB, respectively and $X'Y' \parallel BC$. Let $u := \frac{|AY'|}{|AB|} = \frac{|AX'|}{|AC|}$. The homothety with center A and factor u^{-1} sends X' to C, Y' to B and P' to a point which we denote by P^*. We have

$$\angle BP^*C = \angle Y'P'X' = \angle YPX.$$

Hence, $\angle KIL + \angle YPX = 180°$ is equivalent to the assertion that $P^*BA'C$ is cyclic. We will show this by using the power of the point[1] S defined as the intersection of AI and BC. S lies inside segments $A'P^*$ and BC. So, we need to show

$$|A'S| \cdot |P^*S| = |BS| \cdot |SC|,$$

where all segment lengths are unsigned. Since $ABPC$ is cyclic, we already have

$$|BS| \cdot |SC| = |AS| \cdot |PS|$$

[1]https://calimath.org/wiki/power-of-a-point

by the power of the point S. Let S' be the reflection of S at I. Note that S' lies on $X'Y'$. So, $|AS'| = u|AS|$. We can rewrite $|A'S| = |AS'|$ and $|PS| = |P'S'| = |AP'| + |AS'|$ and by definition, $|P^*S| = |AS| + u^{-1}|AP'|$. It is thus left to show that

$$|AS'| \cdot (|AS| + u^{-1}|AP'|) = |AS| \cdot (|AS'| + |AP'|)$$

or

$$u^{-1}|AS'| = |AS|,$$

which is true. q.e.d.