Problem. A finite set $\mathcal{S}$ of positive integers is called cardinal if $\mathcal{S}$ contains the integer $|\mathcal{S}|$ where $|\mathcal{S}|$ denotes the number of distinct elements in $\mathcal{S}$. Let f be a function from the set of positive integers to itself such that for any cardinal set $\mathcal{S}$, the set $f(\mathcal{S})$ is also cardinal. Here $f(\mathcal{S})$ denotes the set of all integers that can be expressed as $f(a)$ where $a \in \mathcal{S}$. Find all possible values of $f(2024)$.

After trying to solve the problem on your own, you can find a possible solution on the next page.

This problem is related to functional equations. Check out the skillpage[a] to help you solve the problem.

[a] https://calimath.org/skillpages/functional-equations

Problem. A finite set $\mathcal{S}$ of positive integers is called cardinal if $\mathcal{S}$ contains the integer $|\mathcal{S}|$ where $|\mathcal{S}|$ denotes the number of distinct elements in $\mathcal{S}$. Let f be a function from the set of positive integers to itself such that for any cardinal set $\mathcal{S}$, the set $f(\mathcal{S})$ is also cardinal. Here $f(\mathcal{S})$ denotes the set of all integers that can be expressed as $f(a)$ where $a \in \mathcal{S}$. Find all possible values of $f(2024)$.

Proof. We first consider the case that f is unbounded. Let $n \in \mathbb{Z}^+$. Since f is unbounded, we can find integers $m_2, \ldots, m_n$ such that $n < m_2 < \cdots < m_n$ and $\max\{n, f(n)\} < f(m_2) < \cdots < f(m_n)$. $\{n, m_2, \ldots, m_n\}$ is cardinal and therefore $\{f(n), f(m_2), \ldots, f(m_n)\}$ is cardinal, too. Thus, $n \in \{f(n), f(m_2), \ldots, f(m_n)\}$. Since $f(m_i) > n$ for $2 \leq i \leq n$, we conclude that $f(n) = n$. Since n was arbitrary, we get $f(n) = n$ for all $n \in \mathbb{Z}^+$ which is indeed a solution.
Now, we consider the case that f is bounded. Let $t := \max\{f(n) \mid n \in \mathbb{Z}^+\}$. So, $f(n) \in \{1, \ldots, t\}$ for any $n \in \mathbb{Z}^+$. Therefore, we can find an $i \in \{1, \ldots, t\}$ such that $f^{-1}(i) = \{n \in \mathbb{N}^+ \mid f(n) = i\}$ is infinite. We can take any $n \in f^{-1}(i)$ and $n-1$ more elements from $f^{-1}(i)$ to obtain a cardinal set $\mathcal{S}$. Hence, $f(\mathcal{S}) = \{i\}$ is cardinal, implying $i = 1$. In conclusion, $f^{-1}(1)$ is infinite. Consider a new set $\mathcal{S}$ consisting of 2024 and 2023 other numbers from $f^{-1}(1)$. Since $\mathcal{S}$ is cardinal, so is $f(\mathcal{S}) = \{1, f(2024)\}$. So, we must have $f(2024) = a \in \{1, 2\}$. The functions $f = f_a$ with

$$f_a(n) = \begin{cases} a & \text{for } n = 2024, \\ 1 & \text{otherwise.} \end{cases}$$

are indeed solutions for $a \in \{1, 2\}$.
Therefore, the possible values for $f(2024)$ are $1, 2$, and 2024. q.e.d.