Problem. Let $\mathbb{R}_{>0}$ be the set of positive real numbers. Determine all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$x(f(x) + f(y)) \geq (f(f(x)) + y)f(y)$$

for every $x, y \in \mathbb{R}_{>0}$.

After trying to solve the problem on your own, you can find a possible solution on the next page.

This problem is related to functional equations. Check out the skillpage[a] to help you solve the problem.

[a] https://calimath.org/skillpages/functional-equations

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for every $x, y \in \mathbb{R}_{>0}$.

Proof. We will use the notation $f^n(x) = \underbrace{f(f(\dots f}_{n-\text{times}}(x)\dots))$. Let $P(x,y)$ denote the assertion that

$$x(f(x)+f(y)) \geq (f^2(x)+y)f(y).$$

For $f(x) = \frac{c}{x}$ with $c \in \mathbb{R}_{>0}$, the left hand side equals $c + \frac{cx}{y}$ and the right hand side equals $\frac{cx}{y} + c$. Hence, both sides are equal and this is a solution. We will now show that there are no other solutions.

We start with $P(x,x)$ and get

$$2xf(x) \geq f^2(x)f(x) + xf(x).$$

By subtracting $xf(x)$ and dividing by $f(x)$, we get

$$x \geq f(f(x)).$$

$P(f(x), x)$ gives

$$f(x)(f^2(x)+f(x)) \geq (f^3(x)+x)f(x).$$

Hence,

$$f(x) - f^3(x) \geq x - f^2(x).$$

Using this inequality with $f^k(x)$ instead of x, where k is a non-negative integer, we get

$$f^{k+1}(x) - f^{k+3}(x) \geq f^k(x) - f^{k+2}(x). \tag{1}$$

This especially implies

$$f^k(x) - f^{k+2}(x) \geq x - f^2(x) =: \epsilon \geq 0,$$

after applying eq. (1) iteratively. Hence, for $2k$ even, we get

$$f^{2k}(x) \leq f^{2k-2} - \epsilon \leq f^{2k-4}(x) - 2\epsilon \leq \dots \leq f^2(x) - (k-1)\epsilon \leq x - k\epsilon.$$

If $\epsilon > 0$, we can take k large enough, such that $x - k\epsilon < 0$, which is impossible. Thus, we have $\epsilon = 0$ and we conclude $f(f(x)) = x$. $P(x,y)$ now simplifies to

$$xf(x) + xf(y) \geq xf(y) + yf(y)$$

or $xf(x) \geq yf(y)$. By swapping x and y, we get $xf(x) \leq yf(y)$ as well. Hence, $xf(x) = yf(y)$. This implies that $xf(x)$ is constant, so $f(x) = \frac{c}{x}$. q.e.d.