Problem. Let $n \geq 3$ be an integer, and suppose $x_1, x_2, \cdots, x_n$ are positive real numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$x_1^{1-x_2} + x_2^{1-x_3} \cdots + x_{n-1}^{1-x_n} + x_n^{1-x_1} < 2.$$

After trying to solve the problem on your own, you can find a possible solution on the next page.

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Proof. Let $x_{n+1} = x_1$.
For all $1 \leq i \leq n$, we have

$$0 < x_i = 1 - \sum_{\substack{1 \leq j \leq n \\ j \neq i}} x_j < 1.$$

Therefore, it follows that $-1 < -1 + x_i < 0$ and $0 < 1 - x_{i+1} < 1$.
Hence, by Bernoulli's inequality[1]

$$x_i^{1-x_{i+1}} = (1 + (-1 + x_i))^{1-x_{i+1}} \overset{\text{Bernoulli}}{\leq} 1 + (-1 + x_i)(1 - x_{i+1}) = x_i + x_{i+1} - x_i x_{i+1}.$$

Summing this inequality up over all values of i gives

$$\begin{aligned} \sum_{1 \leq i \leq n} x_i^{1-x_{i+1}} &\leq \sum_{1 \leq i \leq n} (x_i + x_{i+1} - x_i x_{i+1}) = 2 \sum_{1 \leq i \leq n} x_i - \sum_{1 \leq i \leq n} x_i x_{i+1} \\ &= 2 - \sum_{1 \leq i \leq n} x_i x_{i+1} < 2, \end{aligned}$$

which is the desired result. q.e.d.

Remark. This problem could also be solved by using the weighted AM-GM inequality[2]. Actually, Bernoulli's inequality is a corollary of weighted AM-GM for exponents $r < 1$. Because $1 + x > 0$ and $1 > 0$ it tells us, that

$$(1 + x)^r = 1^{1-r} \cdot (1 + x)^r \leq (1 - r) \cdot 1 + r \cdot (1 + x) = 1 + rx.$$

[1] https://calimath.org/wiki/bernoullis-inequality
[2] https://calimath.org/wiki/weighted-am-gm-inequality