Problem. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(a^2 + ab + f(b^2)) = af(b) + b^2 + f(a^2) \quad \forall a, b \in \mathbb{R}.$$

After trying to solve the problem on your own, you can find a possible solution on the next page.

This problem is related to functional equations. Check out the skillpage[a] to help you solve the problem.

[a]https://calimath.org/skillpages/functional-equations

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Proof. For $a, b \in \mathbb{R}$, let $P(a, b)$ be the assertion $f(a^2 + ab + f(b^2)) = af(b) + b^2 + f(a^2)$. $P(1, 0)$ and $P(-1, 0)$ tell us

$$f(0) + f(1) = f(1 + f(0)) = -f(0) + f(1)$$

and thus $f(0) = 0$. We have

$$P(0, b) : \qquad f(f(b^2)) = b^2 \tag{1}$$

and thus

$$P(a, -a) : \qquad a^2 \overset{eq.\ (1)}{=} f(f(a^2)) = af(-a) + a^2 + f(a^2).$$

This rearranges to

$$f(a^2) = -af(-a) \tag{2}$$

and thus

$$-a \text{ in eq. (2)} \implies \qquad f(a^2) = af(a), \tag{3}$$

$$\text{eq. (2), eq. (3)} \wedge f(0) = -f(-0) \implies \qquad f(-a) = -f(a). \tag{4}$$

By eq. (1), we get

$$\forall b \in \mathbb{R}, b \geq 0 : \qquad f(f(b)) = f(f(\sqrt{b}^2)) \overset{eq.\ (1)}{=} b.$$

We can plug eq. (4) into the previous equation to obtain

$$\forall b \in \mathbb{R}, b < 0 : \qquad f(f(b)) = f(f(-(-b))) = f(-f(-b)) = -f(f(-b)) = -(-b) = b.$$

In conclusion, we have

$$\forall b \in \mathbb{R} : \qquad f(f(b)) = b. \tag{5}$$

From that equation, we can deduce that f is bijective. Now, we know

$$f(a^2) \overset{eq.\ (3)}{=} af(a) \overset{eq.\ (5)}{=} f(f(a))f(a) \overset{eq.\ (3)}{=} f(f(a)^2).$$

The bijectivity of f tells us that

$$a^2 = f(a)^2$$

or

$$f(a) \in \{-a, a\}. \tag{6}$$

Claim 1. We cannot have $f(a) = a$ and $f(b) = -b$ for $a, b \neq 0$.

Proof. Assume otherwise. Then $f(a^2) = af(a) = a^2$ and $f(b^2) = bf(b) = -b^2$. So $P(a, b)$ becomes

$$f(a^2 + ab - b^2) = a^2 - ab + b^2.$$

Since $a \neq b$, we have

$$a^2 + ab - b^2 = a^2 - ab + b^2 + \underbrace{2b(a - b)}_{\neq 0} \neq a^2 - ab + b^2,$$

$$-(a^2 + ab - b^2) = a^2 - ab + b^2 \underbrace{-2a^2}_{\neq 0} \neq a^2 - ab + b^2.$$

So we have a contradiction. $\square$

Claim 1 together with eq. (6) tell us that the only possible solutions are

$$\boxed{f(a) = a \quad \forall a \in \mathbb{R}} \quad \text{and} \quad \boxed{f(a) = -a \quad \forall a \in \mathbb{R}},$$

which indeed satisfy $P(a, b)$. So these are the solutions to the problem. q.e.d.

Remark 2. A different approach to this problem starts with comparing $P(-a-b, b)$ with $P(a, b)$, which works nicely because the left sides of those equations are equal.