Problem. Determine all prime numbers p and all positive integers x and y satisfying

$$x^3 + y^3 = p(xy + p).$$

After trying to solve the problem on your own, you can find a possible solution on the next page.

Problem. Determine all prime numbers p and all positive integers x and y satisfying

$$x^3 + y^3 = p(xy + p).$$

Proof. We have

$$p(xy + p) = x^3 + y^3 = (x + y)(x^2 - xy + y^2). \tag{1}$$

Assume $p \mid x + y$. If we have $p = x + y$, we would have

$$x^2 - xy + y^2 = xy + p$$

and thus

$$(x - y)^2 = p.$$

This is impossible because a prime number can't be a perfect square. Thus, $p \leq \frac{x+y}{2} \leq xy$. The AM-GM inequality[1] tells us $x^2 - xy + y^2 \geq xy$. This yields

$$(x+y)xy \leq (x+y)(x^2 - xy + y^2) = p(xy + p) \leq \frac{x+y}{2}(xy + \frac{x+y}{2}) \leq= (x+y)xy.$$

Thus we have equality in every bound, which is only possible for $x = y = 1$ and $p = 1$. But $p = 1$ is no prime number, so we get a contradiction. Therefore, we have $\gcd(x + y, p) = 1$ and $p \mid x^2 - xy + y^2$. Thus,

$$p \mid x^2 - xy + y^2 - 3p = (x + y)^2 - 3(xy + p)$$

and

$$\frac{xy + p}{x + y} = \frac{x^2 - xy + y^2}{p} \in \mathbb{N}.$$

This yields

$$x + y \mid xy + p \Longrightarrow x + y \mid (x + y)^2 - 3(xy + p).$$

Together with $\gcd(x + y, p) = 1$, this yields

$$p(x + y) \mid (x + y)^2 - 3(xy + p).$$

Case 1. $(x + y)^2 - 3(xy + p) > 0$.

We have

$$(x + y)^2 > (x + y)^2 - 3(xy + p) \geq p(x + y) \geq \frac{(x + y)^2}{2}.$$

The last inequality follows from $p \geq \frac{x+y}{2}$, which can be proven as above (where we considered the case $p \leq \frac{x+y}{2}$). We get

$$\frac{(x + y)^2 - 3(xy + p)}{p(x + y)} < 2.$$

Thus,

$$(x + y)^2 - 3(xy + p) = p(x + y).$$

This yields

$$xy + p = \frac{(x + y)(x + y - p)}{3}$$

[1] https://calimath.org/wiki/am-gm-inequality

and

$$x^2 - xy + y^2 = p(x + y + 3).$$

Plugging $xy + p$ and $x^2 - xy + y^2$ into **??** yields

$$\frac{x + y - p}{3} = x + y + 3.$$

This is impossible since we work in the positive integers.

Case 2. $(x + y)^2 - 3(xy + p) < 0.$

We have

$$p(x + y) \leq 3(xy + p) - (x + y)^2 \overset{AM-GM}{\leq} 3(xy + p) - 4xy = 3p - xy < 3p.$$

This implies $x + y < 3$, and since they are positive integers, we get $x = y = 1$. **??** implies $p = 1$ or $p = -2$, which are no prime numbers.

Case 3. $(x + y)^2 = 3(xy + p)$

The equation is equivalent to

$$x^2 - xy + y^2 = 3p \tag{2}$$

Multiplying these two equations yields

$$(x + y)^2(x^2 - xy + y^2) = 9p(xy + p).$$

Together with **??** this implies

$$x + y = 9.$$

Checking all possible cases for (x, y) and using **??** to evaluate p gives the solutions

$$(x, y, p) = (8, 1, 19), (1, 8, 19), (7, 2, 13), (2, 7, 13), (5, 4, 7), (4, 5, 7).$$

We can easily check that all these solutions work.

q.e.d.