Problem. Let x, y and $a_0, a_1, a_2, \cdots$ be integers satisfying $a_0 = a_1 = 0$, and

$$a_{n+2} = xa_{n+1} + ya_n + 1$$

for all integers $n \geq 0$. Let p be any prime number. Show that $\gcd(a_p, a_{p+1})$ is either equal to 1 or greater than $\sqrt{p}$.

After trying to solve the problem on your own, you can find a possible solution on the next page.

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Remark 1. I presented a different solution in the video that uses only forward induction and no backward induction.

Proof. Let q be any prime dividing $\gcd(a_p, a_{p+1})$. We will show that $q > \sqrt{p}$, which implies the statement.

Note that $p \geq 2$ and so $0 = a_{p+1} = xa_p + ya_{p-1} + 1 = ya_{p-1} + 1 \in \mathbb{F}_q$ implying $q \nmid y$. Thus, we have

$$a_n = \frac{a_{n+2} - xa_{n+1} - 1}{y} \in \mathbb{F}_q, \tag{1}$$

which tells us that $(a_{n+2}, a_{n+1}) \in \mathbb{F}_q^2$ uniquely determines $a_n \in \mathbb{F}_q$.
We have $|\mathbb{F}_q^2| = q^2$. By the pigeon hole principle, there exists $0 \leq i < j \leq q^2$ with $(a_i, a_{i+1}) = (a_j, a_{j+1}) \in \mathbb{F}_q^2$. Take i, j such that $d := j - i$ is minimal. Using recursion and eq. (1) inductively and by the minimality of d, we get

$$(a_s, a_{s+1}) = (a_t, a_{t+1}) \in \mathbb{F}_q^2 \quad \Longleftrightarrow \quad d \mid t - s.$$

Since $(a_1, a_2) = (0, 1) \neq (0, 0) = (a_0, a_1) \in F_q^2$, we have $d > 1$. From $(a_0, a_1) = (a_p, a_{p+1}) = (0, 0) \in \mathbb{F}_q^2$, we get $d \mid p$, which implies $d = p$ since p is a prime. Thus, $q^2 \geq d = p$. Since p is prime, we cannot have equality. Thus, $q > \sqrt{p}$, as desired. q.e.d.