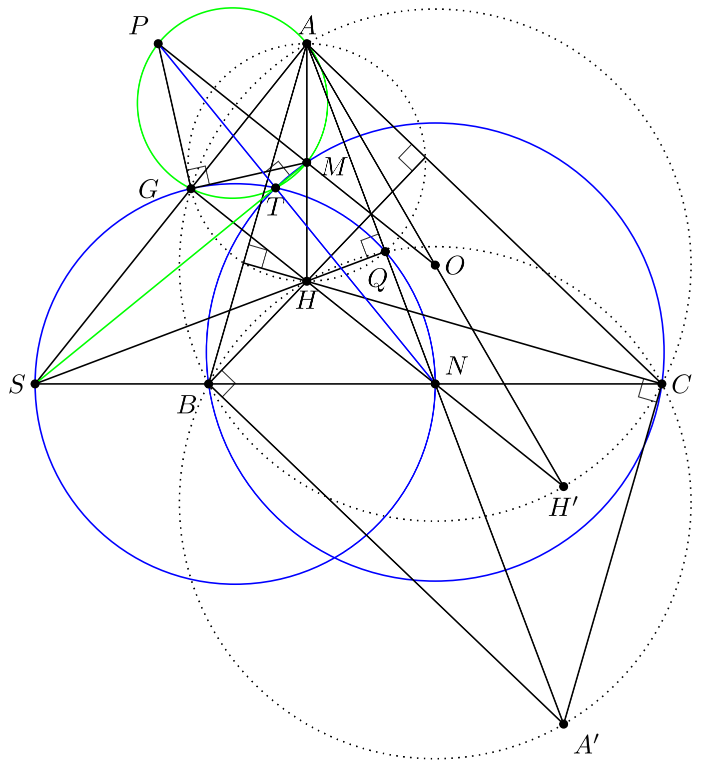
Problem. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.

After trying to solve the problem on your own, you can find a possible solution on the next page.

Problem. Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.



Proof. We use oriented angles modulo 180°. Denote the circumcircle of ABC by Ω and the intersection of AG with BC by S.

Claim 1. S, H, and Q are collinear.

Proof. Denote the reflection of A over N by A'. We get $A'C \parallel AB \perp CH$ and $A'B \parallel AC \perp BH$. Hence, $\measuredangle HCA' = A'BH = 90°$. Moreover, $\measuredangle HQA' = \measuredangle HQA = 90°$. Thus, B, Q, and C all lie on the circle with diameter $A'H$. Hence, $BCQH$ is concyclic. By the radical axis[1] on this circle, Ω, and γ, we get that QH passes through S. □

Claim 2. G, H, and N are collinear.

Proof. Define H' as the reflection of H over N. We get $\measuredangle CH'B = \measuredangle BHC = 180° - \measuredangle BAC = \measuredangle CAB$. Thus, H' lies on Ω. Moreover, $\measuredangle H'AC = \measuredangle H'BC = \measuredangle HCB = 90° - \measuredangle CBA = \measuredangle OAC$. Hence, H' lies on AO, which implies that AH' is a diameter of Ω. Therefore, we get $\measuredangle H'GA = 90° = \measuredangle HGA$. Thus, H', N, H, and G are collinear. □

The claims imply $\measuredangle SGN = 90° = \measuredangle SQN$. Hence, $SGNQ$ is cyclic and it's circumcircle is one of the circles considered in the problem statement. Since M is the midpoint of AH and O is the midpoint of AH', we get that $MO \parallel HH'$. Hence, $\measuredangle AMP = \measuredangle AHG = \measuredangle AGP$. Thus, $AMPG$ is cyclic. Define $T \neq G$ as the second intersection of its circumcirle with the circumcircle

[1] https://calimath.org/wiki/radical-axis

of $SGNQ$. We get

$$\sphericalangle NTG = \sphericalangle NSG = 90^\circ - \sphericalangle SAH = \sphericalangle AHG \overset{M \text{midpoint of } \gamma}{=} \sphericalangle HGM \overset{GH \parallel PM}{=} \sphericalangle PMG = \sphericalangle PTG.$$

This implies P, T, and N are collinear. Since M is the midpoint of γ and P lies on the tangent of γ at G, we know $90^\circ = \sphericalangle MGP = \sphericalangle MTP$. On the other hand, $\sphericalangle STN = \sphericalangle SQN = 90^\circ$ which implies that S, T, and M are collinear. Using power of a point[2] on S, we get

$$SB \cdot SC = SG \cdot SA = ST \cdot SM,$$

where we used signed products. Thus, $BCMT$ is concyclic. Hence, T lies on both of the circumcircles of GNQ and MBC and also on the line PN. q.e.d.

[2]https://calimath.org/wiki/power-of-a-point-theorem